Quelques problèmes de contrôle de micro-nageurs

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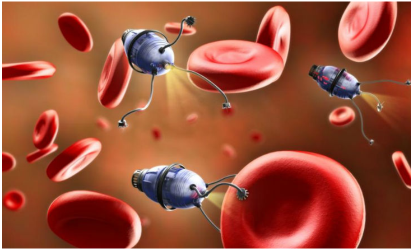
Au programme

- Conditions de contrôlabilité des systèmes contrôle-affines
- Application 1 : contrôle d'un micro-nageur magnétique
- Application 2 : contrôle de deux micro-nageurs



Control theory

Control-affine system

$$\dot{y} = f_0(y) + \sum_{i=1}^{m} f_i(y)u_i \qquad (S)$$

- $y \in \mathbb{R}^n$ is called the **state**
- $f_0, \ldots, f_n$ are real analytic **vector fields** on $\mathbb{R}^n$.
- f_0 is called the **drift**
- $u_1, \ldots, u_m$ are the **controls**

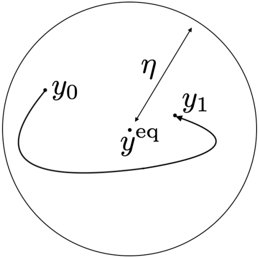
Controllability : given initial and final states y_0 and y_1, and a time T can we find controls u_i such that $y(0) = y_0$ and $y(T) = y_1$?
$\rightarrow$ "drive state y to some place with controls u_i"

Equilibrium : $(y^{\text{eq}}, u^{\text{eq}})$ in $\mathbb{R}^n \times \mathbb{R}^m$ s.t. $f_0(y^{\text{eq}}) + \sum_{i=1}^{m} f_i(y^{\text{eq}})u_i^{\text{eq}} = 0$.

***Local* controllability** : is the system controllable "around" an equilibrium ?

Small-Time Local Controllability (STLC)

For all $\varepsilon > 0$, there exists $\eta > 0$, such that for all y_0, y_1 in $B(y^{\text{eq}}, \eta)$, there exists controls $u(\cdot) = (u_1, \ldots, u_m)$, such that the solution $y(\cdot) : [0, \varepsilon] \to \mathbb{R}^n$ satisfies $y(0) = y_0$, $y(\varepsilon) = y_1$,



and...

α-STLC	STLC
$\exists \alpha > 0, \|u - u^{\text{eq}}\|_{L^\infty[0,\varepsilon]} \leqslant \alpha.$	$\|u - u^{\text{eq}}\|_{L^\infty[0,\varepsilon]} \leqslant \varepsilon\,.$

Conditions on System (S) to be STLC ?

Given two vector fields $f = (f^1, \dots, f^n)$ and $g = (g^1, \dots, g^n)$, their Lie bracket $[f, g]$ is given by :

$$\forall x \in \mathbb{R}^n, [f, g](x) = g'(x)f(x) - f'(x)g(x). \tag{1}$$

In other words, for j in $\{1, \dots, n\}$, the j-th component of $[f, g]$ reads

$$\forall x \in \mathbb{R}^n, [f, g]^j(x) = \sum_{k=1}^{n} f^k(x)\frac{\partial g^j}{\partial x_k}(x) - g^k(x)\frac{\partial f^j}{\partial x_k}(x). \tag{2}$$

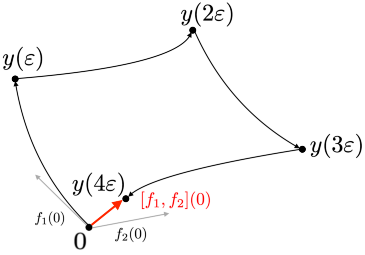
What is the role of Lie brackets in controllability ?

Consider the driftless case with two controls

$$\dot{y} = f_1(y)u_1 + f_2(y)u_2.$$

Let $\varepsilon > 0$ and η_1, η_2 in $\mathbb{R}$. Define the following controls on $[0, 4\varepsilon]$:

$$(u_1(t), u_2(t)) = \begin{cases} (\eta_1, 0) & \text{si } t \in [0, \varepsilon], \\ (0, \eta_2) & \text{si } t \in]\varepsilon, 2\varepsilon], \\ (-\eta_1, 0) & \text{si } t \in]2\varepsilon, 3\varepsilon], \\ (0, -\eta_2) & \text{si } t \in]3\varepsilon, 4\varepsilon]. \end{cases}$$



Then, starting at $y = 0$,

$$y(4\varepsilon) = \eta_1\eta_2\varepsilon^2[f_1, f_2](0) + o(\varepsilon^2).$$

- Reachable directions are (more or less) given by the Lie brackets of the fields f_i.

Conditions of STLC

Define $\text{Lie}(f_0, f_1, \dots, f_m)$ as the space spanned by all iterated Lie brackets of $f_0, \dots, f_m$.

Définition

System (S) satisfies the *Lie algebra rank condition (LARC)* at $(0,0)$ if

$$\text{Span}\,\{g(0), g \in \text{Lie}(f_0, f_1, \dots, f_m)\} = \mathbb{R}^n. \quad (3)$$

Driftless case :

Théorème (Rashevski '38, Chow '39)

Assume $f_0 = 0$*.*

$$(S) \text{ is STLC at } (0,0) \Leftrightarrow (S) \text{ satisfies the LARC at } (0,0).$$

Conditions of STLC

With a drift : not equivalent anymore ! The LARC becomes only a *necessary* condition.

Théorème (Sussmann '73)

(S) *is* α*-STLC at* $(0,0) \Rightarrow (S)$ *satisfies the LARC at* $(0,0)$.

But it is not *sufficient* :

$$\begin{cases} \dot{y}_1 &= y_2^2, \\ \dot{y}_2 &= u. \end{cases} \qquad \text{(Ex)}$$

- (Ex) satisfies the LARC at $(0,0)$
- but (Ex) is not STLC

- No better necessary condition in the general case !

Conditions of STLC : scalar-input systems

We can do a little better in the case $m = 1$, called *scalar-input* :

$$\dot{y} = f_0(y) + f_1(y)u_1. \qquad (S_1)$$

Define

- $\mathcal{S}_k$ the subspace spanned by the Lie brackets of f_0 and f_1 containing at most k times f_1 ;
- S_k the subspace of $\mathbb{R}^n$ spanned by the elements of $\mathcal{S}_k$ evaluated at 0.

Théorème (Sussmann '83)

Assume that, at equilibrium $(0, 0)$ *:*

- *System* (S_1) *satisfies the LARC ;*
- *for all positive integer* k*,*

$$S_{2k} \subset S_{2k-1}. \qquad (4)$$

Then, System (S_1) *is STLC at* $(0, 0)$*.*

What happens when condition (4) does not hold ?

For $k = 1$, it means that

$$S_2 \not\subset S_1.$$

The simplest case is when

$$f_{101}(0) = [f_1, [f_0, f_1]](0) \notin S_1.$$

Théorème (Sussmann '83)

Assume $f_{101}(0)$ does not belong to S_1. Then, System (S_1) is not α-STLC at $(0, 0)$.

- f_{101} is sometimes called a "bad" bracket.

Similar results with non-scalar controls ?

Consider a control-affine system like (S) with $m = 2$:

$$\dot{y} = f_0(y) + f_1(y)u_1 + f_2(y)u_2. \qquad (S_2)$$

Assume that

$$f_0(0) = 0, f_2(0) = 0, \qquad (5)$$

i.e., for all $u_2^{eq} \in \mathbb{R}$, $(0, (0, u_2^{eq}))$ is an equilibrium point :

- the second control u_2 cannot act on the system when at equilibrium position $y = 0$
- role played by this additional control compared to the scalar case ?

Define

- $\mathcal{R}_1$ the subspace spanned by the Lie brackets of f_0, f_1, f_2 containing at most 1 time f_1 ;
- R_1 the subspace of $\mathbb{R}^n$ spanned by the elements of R_1 evaluated at 0.

Théorème (Giraldi, Lissy, M., Pomet '19)

Assume $f_{101}(0) \notin R_1$.

1. *If $f_{101}(0) \in R_1 + Span(f_{121}(0))$ (where $f_{121} = [f_1, [f_2, f_1]]$), let $\beta \in \mathbb{R}$ such that*
$$f_{101}(0) + \beta f_{121}(0) \in R_1.$$
Then, for all $u_2^{eq} \in \mathbb{R}$ such that $u_2^{eq} \neq \beta$, System (S_2) is not STLC at $(0, (0, u_2^{eq}))$.
2. *If $f_{101}(0) \notin R_1 + Span(f_{121}(0))$, then, for all $u_2^{eq} \in \mathbb{R}$, System (S_2) is not α-STLC at $(0, (0, u_2^{eq}))$.*

- The bracket f_{121} can "neutralise the bad bracket" f_{101} if they share a common direction at 0.

Let $k \in \mathbb{N}$ and $I = (i_1, \ldots, i_k)$ in $\{0,1,2\}^k$. Define

$$\int_0^T u_{i_k}(\tau_k) \int_0^{\tau_k} u_{i_{k-1}}(\tau_{k-1}) \int_0^{\tau_{k-1}} \ldots u_{i_2}(\tau_2) \int_0^{\tau_2} u_{i_1}(\tau_1)\mathrm{d}\tau_1 \mathrm{d}\tau_2 \ldots \mathrm{d}\tau_k,$$

with the convention $u_0 = 0$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ a real analytic function defined on a neighbourhood of 0, and controls $u = (u_1, u_2)$ in L^∞. The Chen-Fliess series associated to (S_2), Φ, u and T is defined by

$$\Sigma(u, f, \Phi, T) = \sum_I \left(\int_0^T u_I \right) (f_I \Phi)(0).$$

One has, for T small enough,

$$\Phi(y_u(T)) = \Sigma(u, f, \Phi, T). \tag{6}$$

where y_u is the solution to (S_2) starting at 0 with control u.

$\rightarrow$ this series represents a development of the solution at the origin.

$$\Sigma(u, f, \Phi, T) = \sum_{I} \left(\int_0^T u_I \right) (f_I \Phi)(0).$$

Ideas :

- define a smart choice of function Φ such that
  - if $g \in R_1$, $(g\Phi)(0) = 0$,
  - $(f_{101}\Phi)(0) = 1$.
- Show that the term associated to f_{101}, called P, is positive.
- Show that, under the right hypothesis, the sum of all the other terms is dominated by P, i.e.

$$|\Sigma(u, f, \Phi, T) - P| \leqslant P;$$

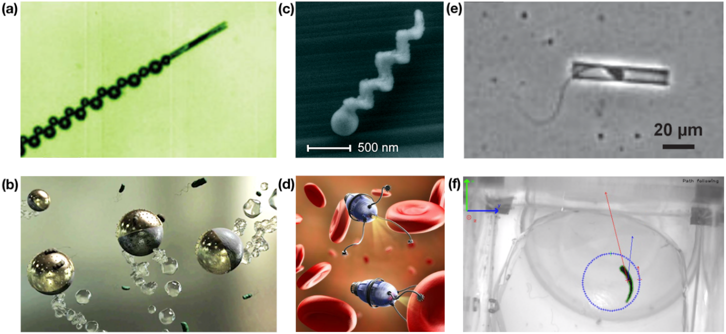
- deduce that $\Phi(y_u(T)) \geqslant 0$ and therefore (S_2) is not STLC.

- Similar results for higher-order brackets (`hal-02178973v3`)
- Application to micro-robots !

Micro-swimming

Context

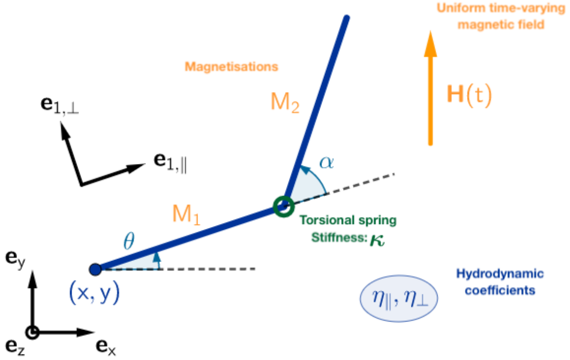
- Propelling micro-robots in fluids
- Study of propulsion with external magnetic field from the control theory point of view
- Biomedical applications



(a) Sovolev *et al.*, *Small* 5(14), 2009. (b) Ghosh, Fischer, *Nano Letters* 9(6), 2009. (c) Peyer *et al.*, *Nanoscale* 5(4), 2013. (e) Magdanz *et al.*, *Adv. Funct. Mater.* 25(18), 2015. (f) El Alaoui-Faris *et al.*, *Phys. Rev. E* 101, 2020.

Application : 2-link magneto-elastic micro-swimmer model

Description of a planar magnetized swimmer, made of two segments S_1 and S_2 :



- The system is described by the **four** state variables $\mathbf{z} = (x, y, \theta, \alpha)$.
- The control is a uniform in space, time-varying magnetic field $\mathbf{H}$. Components in the moving basis $(\mathbf{e}_{1,\parallel}, \mathbf{e}_{1,\perp})$: $(H_\parallel, H_\perp)$.

Equations of the model

1) Internal elastic effect : torque proportional to the shape angle

$$\mathbf{T}^{el} = \kappa\alpha\mathbf{e}_z$$

No elastic effect (*equilibrium*) when the segments are aligned.

2) Magnetic effects : torque proportional to the magnetisations

$$\mathbf{T}_i^m = M_i\mathbf{e}_{i,\parallel} \times \mathbf{H}.$$

The parallel component $H_\parallel$ has no effect on the swimmer when it is at equilibrium !

3) Hydrodynamics effects : density of force at point $\mathbf{x}(s)$ modeled by the Resistive Force Theory [Gray, Hancock 1955] – proportional to the parallel and orthogonal velocity and the coefficients $\eta_\parallel$, $\eta_\perp$.

- Inertia negligible compared to the viscous effects (Low Reynolds number) $\rightarrow$ balance of forces and torques at all times
- Writing balance of forces and torques defines a control-affine system :

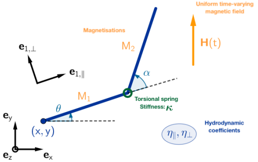
$$\dot{\mathbf{z}} = \mathbf{F}_0(\mathbf{z}) + H_{||}\, \mathbf{F}_1(\mathbf{z}) + H_{\perp}\, \mathbf{F}_2(\mathbf{z})\,.$$

- $H_{||}$ and $H_{\perp}$ seen as the controls

Equilibrium states : $(x, y, \theta, 0)$ (2-link) or $(x, y, \theta, 0, 0)$ (3-link) with magnetic fields $\mathbf{H} = (\alpha, 0)$ with any α ($H_{||}$ has no effect on the swimmer when it is aligned because $\mathbf{F}_1$ vanishes at equilibrium positions)

Are the swimmers STLC ?

Results



Assumption : The magnetizations M_1, and M_2 are such that $M_1 \neq 0$, $M_2 \neq 0$, $M_1 - M_2 \neq 0$ and $M_1 + M_2 \neq 0$.

Proposition

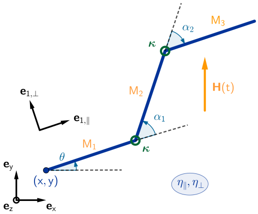
*Then, **the two-link swimmer is STLC at** $(0, (\gamma_2, 0))$ with*

$$\gamma_2 = \kappa \left(\frac{1}{M_1} + \frac{1}{M_2} \right).$$

*Moreover, it is **not STLC at** $(0, (H_\parallel, 0))$ if $H_\parallel \neq \gamma_2$.*

- in particular, not STLC at $(0, (0, 0))$, i.e. with small controls
- the parallel component of the magnetic field has to stay close to the constant value γ_2.
- $R_1(0) = \text{Vect}\,(e_2, e_3, e_4)$;
- $[f_1, [f_0, f_1]](0) = (\dfrac{216(\eta - \xi)\, M_1 M_2\, (M_1 - M_2)}{\ell^8 \eta^3 \xi}, 0, 0, 0)$;
- $[f_1, [f_2, f_1]](0) = (\dfrac{216\kappa(\eta - \xi)(M_1 + M_2)(M_1 - M_2)}{\ell^8 \eta^3 \xi}, 0, 0, 0)$.

Controllability results : 3-link swimmer



Assumption

The magnetizations M_1, M_2 and M_3 are such that

$$\begin{aligned} M_1 - M_3 &\neq 0; \\ (M_1 + M_3 \neq 0 \ \text{ or } \ M_2 &\neq 0); \\ -7M_2^2 + 9M_2(M_1 + M_3) - 5M_1M_3 &\neq 0; \\ P(M_1, M_2, M_3) &\neq 0, \end{aligned} \tag{7}$$

with

$$P(x, y, z) = 49y^3 - 91y^2(x+z) + 36y(x+z)^2 - (45y + 65(x+z))xz.$$

Proposition

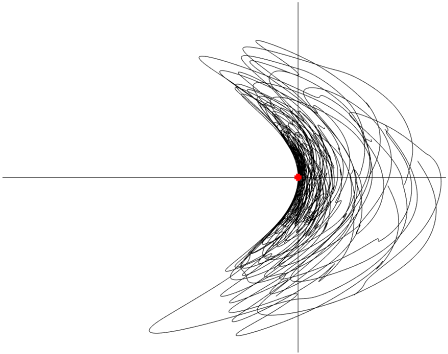
Under Assumption 2, ***the three-link swimmer is STLC at*** $(0, (\gamma_3, 0))$ *with*

$$\gamma_3 = \kappa \frac{17(M_1 + M_3) - 16M_2}{-7M_2^2 + 9M_2(M_1 + M_3) - 5M_1M_3}.$$

Moreover, it is ***not STLC at*** $(0, (\alpha, 0))$ *with* $\alpha \neq \gamma_3$.

Numerical simulations

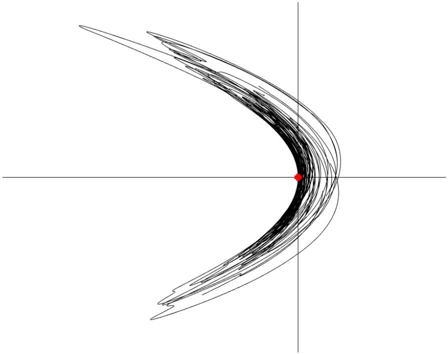
- Start from the equilibrium position $(0, 0, 0, 0)$ and calculate the trajectory for controls varying “around" the reference control $(\beta, 0)$ with $\beta \in \mathbb{R}$.
- Plot the evolution of the extremity (x, y).



$\beta = 0.5\gamma_2$

Numerical simulations

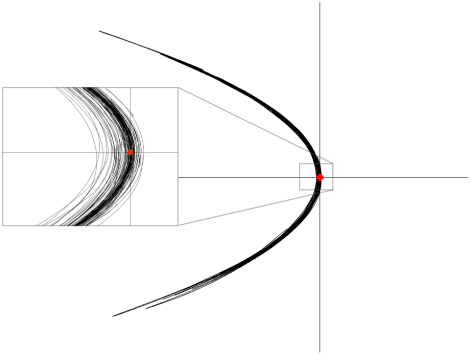
- Start from the equilibrium position $(0, 0, 0, 0)$ and calculate the trajectory for controls varying "around" the reference control $(\beta, 0)$ with $\beta \in \mathbb{R}$.
- Plot the evolution of the extremity (x, y).



$\beta = 0.9\gamma_2$

Numerical simulations

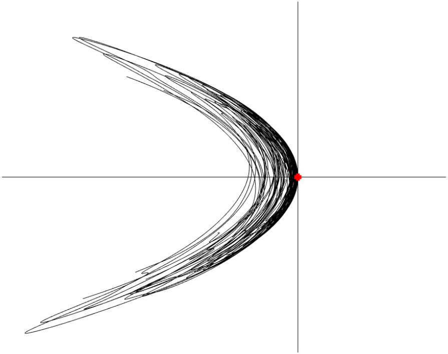
- Start from the equilibrium position $(0, 0, 0, 0)$ and calculate the trajectory for controls varying "around" the reference control $(\beta, 0)$ with $\beta \in \mathbb{R}$.
- Plot the evolution of the extremity (x, y).



$$\beta = \gamma_2$$

Numerical simulations

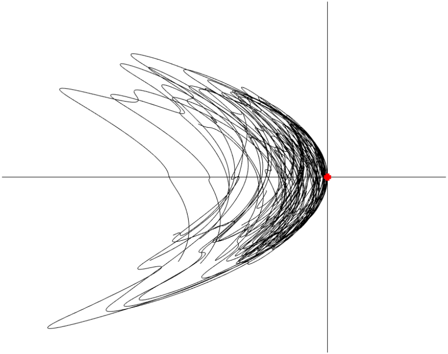
- Start from the equilibrium position $(0,0,0,0)$ and calculate the trajectory for controls varying "around" the reference control $(\beta, 0)$ with $\beta \in \mathbb{R}$.
- Plot the evolution of the extremity (x, y).



$\beta = 1.1\gamma_2$

Numerical simulations

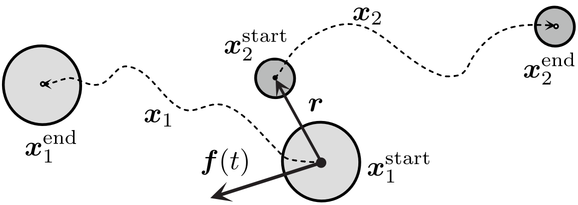
- Start from the equilibrium position $(0, 0, 0, 0)$ and calculate the trajectory for controls varying "around" the reference control $(\beta, 0)$ with $\beta \in \mathbb{R}$.
- Plot the evolution of the extremity (x, y).



$\beta = 1.25\gamma_2$

Contrôle de plusieurs micro-nageurs

Modèle



- Deux particules sphériques
- Application d'une force **f** sur l'une
- Contrôle de la position des deux en même temps

- Régime de Stokes → linéarité

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{M}\\ \boldsymbol{N}\end{bmatrix}\boldsymbol{f}, \tag{8}$$

que l'on récrit sous la forme d'un système contrôle-affine (sans dérive) :

$$\begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{r}\end{bmatrix} = \begin{bmatrix} M_{\|}(r)\frac{\boldsymbol{r}\boldsymbol{r}^T}{\|\boldsymbol{r}\|^2} + M_{\perp}(r)\left(\boldsymbol{I} - \frac{\boldsymbol{r}\boldsymbol{r}^T}{\|\boldsymbol{r}\|^2}\right)\\ -m_{\|}(r)\frac{\boldsymbol{r}\boldsymbol{r}^T}{\|\boldsymbol{r}\|^2} - m_{\perp}(r)\left(\boldsymbol{I} - \frac{\boldsymbol{r}\boldsymbol{r}^T}{\|\boldsymbol{r}\|^2}\right)\end{bmatrix}\boldsymbol{f} = f_x\mathbf{g}_x + f_y\mathbf{g}_y + f_z\mathbf{g}_z. \tag{9}$$

où $M_{\|}, M_{\perp}, m_{\|}, m_{\perp}$ sont les **coefficients de mobilité**.

Contrôlabilité

Pas de dérive, donc il suffit de vérifier la LARC et d'appliquer le théorème de Rashevski-Chow. On considère la matrice

$$\mathbf{C} = \left[\boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{g}_z, [\boldsymbol{g}_x, \boldsymbol{g}_y], [\boldsymbol{g}_x, \boldsymbol{g}_z], [\boldsymbol{g}_y, [\boldsymbol{g}_x, \boldsymbol{g}_y]]\right] . \qquad (10)$$

Son déterminant est

$$\frac{{m_\parallel}^4 {m_\perp}^6}{r} \left[\frac{1}{r}\left(\frac{M_\parallel}{m_\parallel} - \frac{M_\perp}{m_\perp}\right) - \left(\frac{M_\perp}{m_\perp}\right)'\right]^3 , \qquad (11)$$

et on obtient donc la condition de contrôlabilité

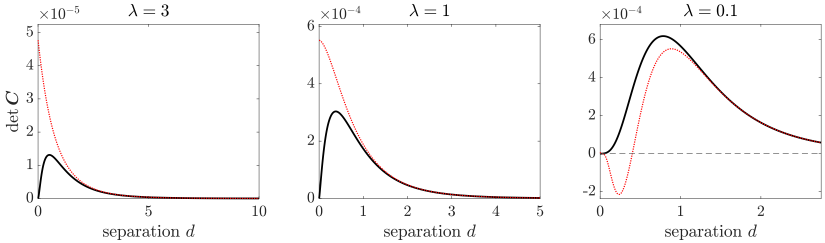
Proposition

Le système est contrôlable en r si (et seulement si)

$$\frac{1}{r}\left(\frac{M_\parallel}{m_\parallel} - \frac{M_\perp}{m_\perp}\right) - \left(\frac{M_\perp}{m_\perp}\right)' \neq 0.$$

Calcul du déterminant

- Les coefficients de mobilité peuvent être approchés par des séries en $\frac{1}{r^n}$
- rouge : ordre $\frac{1}{r}$, noir : ordre $\frac{1}{r^{200}}$



- contrôlable partout !
- en utilisant les crochets de **C** comme des “vecteurs de base”, on peut expliciter des contrôles qui réalisent un certain objectif $\rightarrow$ planification de trajectoires

Références

- Conditions de contrôlabilité locale de systèmes contrôle-affines avec dérive

L. Giraldi, P. Lissy, C. M. and J.-B. Pomet, Necessary conditions for local controllability of a particular class of systems with two scalar controls, hal :02178973v3

- Application à des nageurs magnétiques

C. M., Local controllability of a magnetized Purcell's swimmer, IEEE Control Systems Letters, vol.3, no.3, pp. 637-642, May 2019.

- Etude de la contrôlabilité de plusieurs particules et planification de trajectoire

B. J. Walker, K. Ishimoto, E. A. Gaffney, C. M., The control of particles in the Stokes limit, arxiv :2105.13550

Merci pour votre attention !

What happens when $f_{101}(0)$ does belong to S_1 ?

In the **scalar case**, the next simplest case is when

$$f_{01,001}(0) = [[f_0, f_1], [f_0, [f_0, f_1]]](0) \notin S_1(0).$$

$\rightarrow$ In this case, System (S_1) may or may not be STLC.

For f, g two vector fields, let $\mathrm{ad}_f^0 g = g$ and for all positive integer k,

$$\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g].$$

Théorème (Kawski '88)

Let $S' = Span\{ad_{f_0}^k(ad_{f_1}^3 f_0), k \in \mathbb{N}\}$. If $f_{01,001}(0) \notin S_1(0) + S'(0)$, then ($S_1$) is not STLC at $(0,0)$.

Théorème (Beauchard, Marbach '17)

If $f_{01,001}(0) \notin S_1(0)$, then it is not $W^{1,\infty}$-STLC at $(0,0)$.

Two controls ?

- **seven other brackets** of order 5 are involved, via the map

$$D_{u_2^{\mathrm{eq}}}(\lambda_1, \lambda_2) = \lambda_1^2(f_{01,001}(0) - u_2^{\mathrm{eq}} f_{01,201}(0)) + \lambda_2^2(f_{21,021}(0) - u_2^{\mathrm{eq}} f_{21,221}(0)) \\ -\lambda_1\lambda_2(f_{21,001}(0) + f_{01,021}(0) - u_2^{\mathrm{eq}}(f_{21,201}(0) + f_{01,221}(0))),$$

Théorème

Assume that $f_{101}(0), f_{121}(0)$ and $f_{21,01}(0)$ belong to R_1 and let $u_2^{eq} \in \mathbb{R}$. Then :

1. *If there exists a linear form φ on $\mathbb{R}^n$ such that*

$$R_1 \subset \ker \varphi, \tag{12}$$

$$(\lambda_1, \lambda_2) \mapsto \langle \varphi \, , \, D_{u_2^{eq}}(\lambda_1, \lambda_2) \rangle \text{ is positive definite,} \tag{13}$$

then system (S_2) is not $(\mathrm{W}^{1,\infty}, \mathrm{L}^{\infty})$-STLC at $(0, (0, u_2^{eq}))$.

2. *If there exists a linear form ψ on $\mathbb{R}^n$ such that*

$$R' \subset \ker \psi, \tag{14}$$

$$(\lambda_1, \lambda_2) \mapsto \langle \psi \, , \, D_{u_2^{eq}}(\lambda_1, \lambda_2) \rangle \text{ is positive definite,} \tag{15}$$

then, system (S_2) is not $(\mathrm{W}^{1,\infty}, B)$-STLC at $(0, (0, u_2^{eq}))$.

- structurally similar to the one for f_{101} but more technical